

C4 INTEGRATION

Answers - Worksheet E

- 1 a** $u = x^2 + 1 \therefore \frac{du}{dx} = 2x$

$$\int 2x(x^2 - 1)^3 dx = \int u^3 du$$

$$= \frac{1}{4}u^4 + c$$

$$= \frac{1}{4}(x^2 + 1)^4 + c$$
- b** $u = \sin x \therefore \frac{du}{dx} = \cos x$

$$\int \sin^4 x \cos x dx = \int u^4 du$$

$$= \frac{1}{5}u^5 + c$$

$$= \frac{1}{5}\sin^5 x + c$$
- c** $u = 2 + x^3 \therefore \frac{du}{dx} = 3x^2$

$$\int 3x^2(2 + x^3)^2 dx = \int u^2 du$$

$$= \frac{1}{3}u^3 + c$$

$$= \frac{1}{3}(2 + x^3)^3 + c$$
- d** $u = x^2 \therefore \frac{du}{dx} = 2x$

$$\int 2xe^{-x^2} dx = \int e^u du$$

$$= e^u + c$$

$$= e^{-x^2} + c$$
- e** $u = x^2 + 3 \therefore \frac{du}{dx} = 2x$

$$\int \frac{x}{(x^2 + 3)^4} dx = \int \frac{1}{2}u^{-4} du$$

$$= -\frac{1}{6}u^{-3} + c$$

$$= -\frac{1}{6(x^2 + 3)^3} + c$$
- f** $u = \cos 2x \therefore \frac{du}{dx} = -2 \sin 2x$

$$\int \sin 2x \cos^3 2x dx = \int -\frac{1}{2}u^3 du$$

$$= -\frac{1}{8}u^4 + c$$

$$= -\frac{1}{8}\cos^4 2x + c$$
- g** $u = x^2 - 2 \therefore \frac{du}{dx} = 2x$

$$\int \frac{3x}{x^2 - 2} dx = \int \frac{3}{2u} du$$

$$= \frac{3}{2} \ln |u| + c$$

$$= \frac{3}{2} \ln |x^2 - 2| + c$$
- h** $u = 1 - x^2 \therefore \frac{du}{dx} = -2x$

$$\int x\sqrt{1-x^2} dx = \int -\frac{1}{2}u^{\frac{1}{2}} du$$

$$= -\frac{1}{3}u^{\frac{3}{2}} + c$$

$$= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + c$$
- i** $u = \sec x \therefore \frac{du}{dx} = \sec x \tan x$

$$\int \sec^3 x \tan x dx = \int u^2 du$$

$$= \frac{1}{3}u^3 + c$$

$$= \frac{1}{3}\sec^3 x + c$$
- j** $u = x^2 + 2x \therefore \frac{du}{dx} = 2x + 2$

$$\int (x+1)(x^2 + 2x)^3 dx = \int \frac{1}{2}u^3 du$$

$$= \frac{1}{8}u^4 + c$$

$$= \frac{1}{8}(x^2 + 2x)^4 + c$$
- 2 a i** $u = 3$
ii $u = 4$
- b** $u = x^2 + 3 \therefore \frac{du}{dx} = 2x$

$$\int_0^1 2x(x^2 + 3)^2 dx = \int_3^4 u^2 \times \frac{du}{dx} dx$$

$$= \int_3^4 u^2 du$$
- c** $\int_0^1 2x(x^2 + 3)^2 dx = \int_3^4 u^2 du$

$$= \left[\frac{1}{3}u^3 \right]_3^4$$

$$= \frac{64}{3} - 9 = 12\frac{1}{3}$$

$$3 \quad \mathbf{a} \quad u = x^2 - 3 \quad \therefore \frac{du}{dx} = 2x$$

$$x = 1 \Rightarrow u = -2$$

$$x = 2 \Rightarrow u = 1$$

$$\begin{aligned} \int_1^2 x(x^2 - 3)^3 dx &= \int_{-2}^1 \frac{1}{2} u^3 du \\ &= \left[\frac{1}{8} u^4 \right]_{-2}^1 \\ &= \frac{1}{8} (1 - 16) \\ &= -\frac{15}{8} \end{aligned}$$

$$\mathbf{c} \quad u = x^2 + 1 \quad \therefore \frac{du}{dx} = 2x$$

$$x = 0 \Rightarrow u = 1$$

$$x = 3 \Rightarrow u = 10$$

$$\begin{aligned} \int_0^3 \frac{4x}{x^2+1} dx &= \int_1^{10} \frac{2}{u} du \\ &= [2 \ln |u|]_1^{10} \\ &= 2 \ln 10 - 0 \\ &= 2 \ln 10 \end{aligned}$$

$$\mathbf{e} \quad u = x^2 - 3 \quad \therefore \frac{du}{dx} = 2x$$

$$x = 2 \Rightarrow u = 1$$

$$x = 3 \Rightarrow u = 6$$

$$\begin{aligned} \int_2^3 \frac{x}{\sqrt{x^2-3}} dx &= \int_1^6 \frac{1}{2} u^{-\frac{1}{2}} du \\ &= [u^{\frac{1}{2}}]_1^6 \\ &= \sqrt{6} - 1 \end{aligned}$$

$$\mathbf{g} \quad u = 1 + e^{2x} \quad \therefore \frac{du}{dx} = 2e^{2x}$$

$$x = 0 \Rightarrow u = 2$$

$$x = 1 \Rightarrow u = 1 + e^2$$

$$\begin{aligned} \int_0^1 e^{2x}(1 + e^{2x})^3 dx &= \int_2^{1+e^2} \frac{1}{2} u^3 du \\ &= \left[\frac{1}{8} u^4 \right]_2^{1+e^2} \\ &= \frac{1}{8} [(1 + e^2)^4 - 16] \\ &= \frac{1}{8} (1 + e^2)^4 - 2 \end{aligned}$$

$$\mathbf{b} \quad u = \sin x \quad \therefore \frac{du}{dx} = \cos x$$

$$x = 0 \Rightarrow u = 0$$

$$x = \frac{\pi}{6} \Rightarrow u = \frac{1}{2}$$

$$\begin{aligned} \int_0^{\frac{\pi}{6}} \sin^3 x \cos x dx &= \int_0^{\frac{1}{2}} u^3 du \\ &= \left[\frac{1}{4} u^4 \right]_0^{\frac{1}{2}} \\ &= \frac{1}{4} \left(\frac{1}{16} - 0 \right) \\ &= \frac{1}{64} \end{aligned}$$

$$\mathbf{d} \quad u = \tan x \quad \therefore \frac{du}{dx} = \sec^2 x$$

$$x = -\frac{\pi}{4} \Rightarrow u = -1$$

$$x = \frac{\pi}{4} \Rightarrow u = 1$$

$$\begin{aligned} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \sec^2 x dx &= \int_{-1}^1 u^2 du \\ &= \left[\frac{1}{3} u^3 \right]_{-1}^1 \\ &= \frac{1}{3} [1 - (-1)] \\ &= \frac{2}{3} \end{aligned}$$

$$\mathbf{f} \quad u = x^3 + 2 \quad \therefore \frac{du}{dx} = 3x^2$$

$$x = -2 \Rightarrow u = -6$$

$$x = -1 \Rightarrow u = 1$$

$$\begin{aligned} \int_{-2}^{-1} x^2(x^3 + 2)^2 dx &= \int_{-6}^1 \frac{1}{3} u^2 du \\ &= \left[\frac{1}{9} u^3 \right]_{-6}^1 \\ &= \frac{1}{9} [1 - (-216)] \\ &= 24\frac{1}{9} \end{aligned}$$

$$\mathbf{h} \quad u = x^2 - 4x \quad \therefore \frac{du}{dx} = 2x - 4$$

$$x = 3 \Rightarrow u = -3$$

$$x = 5 \Rightarrow u = 5$$

$$\begin{aligned} \int_3^5 (x-2)(x^2-4x)^2 dx &= \int_{-3}^5 \frac{1}{2} u^2 du \\ &= \left[\frac{1}{6} u^3 \right]_{-3}^5 \\ &= \frac{1}{6} [125 - (-27)] \\ &= 25\frac{1}{3} \end{aligned}$$

$$4 \quad \mathbf{a} \quad u = 4 - x^2 \quad \therefore \frac{du}{dx} = -2x$$

$$x = 0 \Rightarrow u = 4$$

$$x = 2 \Rightarrow u = 0$$

$$\begin{aligned} \int_0^2 x(4-x^2)^3 dx &= \int_4^0 u^3 \times \left(-\frac{1}{2} \frac{du}{dx}\right) du \\ &= \int_0^4 \frac{1}{2} u^3 du \end{aligned}$$

$$\begin{aligned} \mathbf{b} &= \left[\frac{1}{8} u^4\right]_0^4 \\ &= \frac{1}{8} (256 - 0) \\ &= 32 \end{aligned}$$

$$5 \quad \mathbf{a} \quad u = 2 - x^2 \quad \therefore \frac{du}{dx} = -2x$$

$$x = 0 \Rightarrow u = 2$$

$$x = 1 \Rightarrow u = 1$$

$$\begin{aligned} \int_0^1 x e^{2-x^2} dx &= \int_2^1 -\frac{1}{2} e^u du \\ &= \int_1^2 \frac{1}{2} e^u du \\ &= \left[\frac{1}{2} e^u\right]_1^2 \\ &= \frac{1}{2} (e^2 - e) \\ &= \frac{1}{2} e(e - 1) \end{aligned}$$

$$\mathbf{b} \quad u = 1 + \cos x \quad \therefore \frac{du}{dx} = -\sin x$$

$$x = 0 \Rightarrow u = 2$$

$$x = \frac{\pi}{2} \Rightarrow u = 1$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos x} dx &= \int_2^1 -\frac{1}{u} du \\ &= \int_1^2 \frac{1}{u} du \\ &= [\ln |u|]_1^2 \\ &= \ln 2 - 0 \\ &= \ln 2 \end{aligned}$$

$$6 \quad \mathbf{a} \quad u = \sin x \quad \therefore \frac{du}{dx} = \cos x$$

$$\begin{aligned} \int \cot x dx &= \int \frac{\cos x}{\sin x} dx \\ &= \int \frac{1}{u} \times \frac{du}{dx} dx \\ &= \int \frac{1}{u} du \\ &= \ln |u| + c \\ &= \ln |\sin x| + c \end{aligned}$$

$$\mathbf{b} \quad \int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

$$u = \cos x \quad \therefore \frac{du}{dx} = -\sin x$$

$$\begin{aligned} \int \frac{\sin x}{\cos x} dx &= \int \frac{1}{u} \times \left(-\frac{du}{dx}\right) dx \\ &= \int -\frac{1}{u} du \\ &= -\ln |u| + c \\ &= -\ln |\cos x| + c \\ &= \ln (|\cos x|)^{-1} + c \\ &= \ln |\sec x| + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} &= \left[\frac{1}{2} \ln |\sec 2x|\right]_0^{\frac{\pi}{6}} \\ &= \frac{1}{2} (\ln 2 - 0) \\ &= \frac{1}{2} \ln 2 \end{aligned}$$

$$\begin{array}{lll}
 7 \quad \mathbf{a} & = \frac{1}{4}(x^3 - 2)^4 + c & \mathbf{b} = e^{\sin x} + c & \mathbf{c} = \frac{1}{2} \int \frac{2x}{x^2+1} dx \\
 & & & = \frac{1}{2} \ln |x^2 + 1| + c \\
 & & & [= \frac{1}{2} \ln (x^2 + 1) + c] \\
 \\
 \mathbf{d} & = \frac{1}{3}(x^2 + 3x)^3 + c & \mathbf{e} = \frac{1}{2} \int 2x(x^2 + 4)^{\frac{1}{2}} dx & \mathbf{f} = -\int \cot^3 x (-\operatorname{cosec}^2 x) dx \\
 & & = \frac{1}{2} \times \frac{2}{3}(x^2 + 4)^{\frac{3}{2}} + c & = -\frac{1}{4} \cot^4 x + c \\
 & & = \frac{1}{3}(x^2 + 4)^{\frac{3}{2}} + c & \\
 \\
 \mathbf{g} & = \ln |1 + e^x| + c & \mathbf{h} = \frac{1}{2} \int \frac{2 \cos 2x}{3 + \sin 2x} dx & \mathbf{i} = \frac{1}{4} \int \frac{4x^3}{(x^4 - 2)^2} + c \\
 & [= \ln (1 + e^x) + c] & = \frac{1}{2} \ln |3 + \sin 2x| + c & = \frac{1}{4} \times [-(x^4 - 2)^{-1}] + c \\
 & & & = -\frac{1}{4(x^4 - 2)} + c \\
 \\
 \mathbf{j} & = \frac{1}{4}(\ln x)^4 + c & \mathbf{k} = \frac{2}{3} \int \frac{3}{2} x^{\frac{1}{2}} (1 + x^{\frac{3}{2}})^2 dx & \mathbf{l} = -\frac{1}{2} \int -2x(5 - x^2)^{-\frac{1}{2}} dx \\
 & & = \frac{2}{3} \times \frac{1}{3} (1 + x^{\frac{3}{2}})^3 + c & = -\frac{1}{2} \times 2(5 - x^2)^{\frac{1}{2}} + c \\
 & & = \frac{2}{9} (1 + x^{\frac{3}{2}})^3 + c & = -\sqrt{5 - x^2} + c
 \end{array}$$

$$\begin{array}{ll}
 8 \quad \mathbf{a} & = -\int_0^{\frac{\pi}{2}} (-\sin x)(1 + \cos x)^2 dx \\
 & = -\left[\frac{1}{3}(1 + \cos x)^3\right]_0^{\frac{\pi}{2}} \\
 & = -\frac{1}{3}(1 - 8) \\
 & = \frac{7}{3} \\
 \\
 \mathbf{b} & = -\frac{1}{2} \int_{-1}^0 \frac{-2e^{2x}}{2 - e^{2x}} dx \\
 & = -\frac{1}{2} [\ln |2 - e^{2x}|]_{-1}^0 \\
 & = -\frac{1}{2} [0 - \ln (2 - e^{-2})] \\
 & = \frac{1}{2} \ln (2 - e^{-2}) \\
 \\
 \mathbf{c} & = -\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (-\cot x \operatorname{cosec} x) \operatorname{cosec}^3 x dx \\
 & = -\left[\frac{1}{4} \operatorname{cosec}^4 x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} \\
 & = -\frac{1}{4}(4 - 16) \\
 & = 3 \\
 \\
 \mathbf{d} & = \frac{1}{2} \int_2^4 \frac{2x+2}{x^2+2x+8} dx \\
 & = \frac{1}{2} [\ln |x^2 + 2x + 8|]_2^4 \\
 & = \frac{1}{2} (\ln 32 - \ln 16) \\
 & = \frac{1}{2} \ln 2
 \end{array}$$

$$\begin{array}{l}
 9 \quad u = x + 1 \quad \therefore x = u - 1, \quad \frac{du}{dx} = 1 \\
 \int x(x + 1)^3 dx = \int (u - 1)u^3 du \\
 = \int (u^4 - u^3) du \\
 = \frac{1}{5}u^5 - \frac{1}{4}u^4 + c \\
 = \frac{1}{5}(x + 1)^5 - \frac{1}{4}(x + 1)^4 + c \\
 = \frac{1}{20}(x + 1)^4[4(x + 1) - 5] + c \\
 = \frac{1}{20}(4x - 1)(x + 1)^4 + c
 \end{array}$$

$$10 \quad \mathbf{a} \quad u = 2x - 1 \quad \therefore x = \frac{1}{2}(u + 1), \quad \frac{du}{dx} = 2$$

$$\begin{aligned} \int x(2x-1)^4 dx &= \int \frac{1}{2}(u+1)u^4 \times \frac{1}{2} du \\ &= \frac{1}{4} \int (u^5 + u^4) du \\ &= \frac{1}{4} \left(\frac{1}{6}u^6 + \frac{1}{5}u^5 \right) + c \\ &= \frac{1}{4} \left[\frac{1}{6}(2x-1)^6 + \frac{1}{5}(2x-1)^5 \right] + c \\ &= \frac{1}{120} (2x-1)^5 [5(2x-1) + 6] + c \\ &= \frac{1}{120} (10x+1)(2x-1)^5 + c \end{aligned}$$

$$\mathbf{c} \quad x = \sin u \quad \therefore \frac{dx}{du} = \cos u$$

$$\begin{aligned} \int \frac{1}{(1-x^2)^{\frac{3}{2}}} dx &= \int \frac{1}{\cos^3 u} \times \cos u du \\ &= \int \sec^2 u du \\ &= \tan u + c \\ &= \frac{\sin u}{\cos u} + c \\ &= \frac{x}{\sqrt{1-x^2}} + c \end{aligned}$$

$$\mathbf{e} \quad u = 2x + 3 \quad \therefore x = \frac{1}{2}u - \frac{3}{2}, \quad \frac{du}{dx} = 2$$

$$\begin{aligned} \int (x+1)(2x+3)^3 dx &= \int \left(\frac{1}{2}u - \frac{1}{2} \right) u^3 \times \frac{1}{2} du \\ &= \frac{1}{4} \int (u^4 - u^3) du \\ &= \frac{1}{4} \left(\frac{1}{5}u^5 - \frac{1}{4}u^4 \right) + c \\ &= \frac{1}{4} \left[\frac{1}{5}(2x+3)^5 - \frac{1}{4}(2x+3)^4 \right] + c \\ &= \frac{1}{80} (2x+3)^4 [4(2x+3) - 5] + c \\ &= \frac{1}{80} (8x+7)(2x+3)^4 + c \end{aligned}$$

$$\mathbf{b} \quad u^2 = 1 - x \quad \therefore x = 1 - u^2, \quad \frac{dx}{du} = -2u$$

$$\begin{aligned} \int x\sqrt{1-x} dx &= \int (1-u^2)u \times (-2u) du \\ &= 2 \int (u^4 - u^2) du \\ &= 2 \left(\frac{1}{5}u^5 - \frac{1}{3}u^3 \right) + c \\ &= 2 \left[\frac{1}{5}(1-x)^{\frac{5}{2}} - \frac{1}{3}(1-x)^{\frac{3}{2}} \right] + c \\ &= \frac{2}{15} (1-x)^{\frac{3}{2}} [3(1-x) - 5] + c \\ &= -\frac{2}{15} (2+3x)(1-x)^{\frac{3}{2}} + c \end{aligned}$$

$$\mathbf{d} \quad x = u^2 \quad \therefore \frac{dx}{du} = 2u$$

$$\begin{aligned} \int \frac{1}{\sqrt{x-1}} dx &= \int \frac{1}{u-1} \times 2u du \\ &= \int \frac{2(u-1)+2}{u-1} du \\ &= \int \left(2 + \frac{2}{u-1} \right) du \\ &= 2u + 2 \ln |u-1| + c \\ &= 2\sqrt{x} + 2 \ln |\sqrt{x}-1| + c \end{aligned}$$

$$\mathbf{f} \quad u^2 = x - 2 \quad \therefore x = u^2 + 2, \quad \frac{dx}{du} = 2u$$

$$\begin{aligned} \int \frac{x^2}{\sqrt{x-2}} dx &= \int \frac{(u^2+2)^2}{u} \times 2u du \\ &= 2 \int (u^4 + 4u^2 + 4) du \\ &= 2 \left(\frac{1}{5}u^5 + \frac{4}{3}u^3 + 4u \right) + c \\ &= 2 \left[\frac{1}{5}(x-2)^{\frac{5}{2}} + \frac{4}{3}(x-2)^{\frac{3}{2}} + 4(x-2)^{\frac{1}{2}} \right] + c \\ &= \frac{2}{15} (x-2)^{\frac{1}{2}} [3(x-2)^2 + 20(x-2) + 60] + c \\ &= \frac{2}{15} (3x^2 + 8x + 32)(x-2)^{\frac{1}{2}} + c \end{aligned}$$

$$11 \quad \mathbf{a} \quad x = \sin u \quad \therefore \frac{dx}{du} = \cos u$$

$$x = 0 \Rightarrow u = 0$$

$$x = \frac{1}{2} \Rightarrow u = \frac{\pi}{6}$$

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx = \int_0^{\frac{\pi}{6}} \frac{1}{\cos u} \times \cos u du$$

$$= \int_0^{\frac{\pi}{6}} du$$

$$= [u]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{6} - 0$$

$$= \frac{\pi}{6}$$

$$\mathbf{b} \quad u = 2 - x \quad \therefore x = 2 - u, \quad \frac{du}{dx} = -1$$

$$x = 0 \Rightarrow u = 2$$

$$x = 2 \Rightarrow u = 0$$

$$\int_0^2 x(2-x)^3 dx = \int_2^0 (2-u)u^3 \times (-1) du$$

$$= \int_0^2 (2u^3 - u^4) du$$

$$= \left[\frac{1}{2}u^4 - \frac{1}{5}u^5 \right]_0^2$$

$$= \left(8 - \frac{32}{5} \right) - (0)$$

$$= \frac{8}{5}$$

$$\mathbf{c} \quad x = 2 \sin u \quad \therefore \frac{dx}{du} = 2 \cos u$$

$$x = 0 \Rightarrow u = 0$$

$$x = 1 \Rightarrow u = \frac{\pi}{6}$$

$$\int_0^1 \sqrt{4-x^2} dx$$

$$= \int_0^{\frac{\pi}{6}} 2 \cos u \times 2 \cos u du$$

$$= \int_0^{\frac{\pi}{6}} 4 \cos^2 u du$$

$$= \int_0^{\frac{\pi}{6}} (2 + 2 \cos 2u) du$$

$$= [2u + \sin 2u]_0^{\frac{\pi}{6}}$$

$$= \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) - (0)$$

$$= \frac{1}{6}(2\pi + 3\sqrt{3})$$

$$\mathbf{d} \quad x = 3 \tan u \quad \therefore \frac{dx}{du} = 3 \sec^2 u$$

$$x = 0 \Rightarrow u = 0$$

$$x = 3 \Rightarrow u = \frac{\pi}{4}$$

$$\int_0^3 \frac{x^2}{x^2+9} dx = \int_0^{\frac{\pi}{4}} \frac{9 \tan^2 u}{9 \sec^2 u} \times 3 \sec^2 u du$$

$$= 3 \int_0^{\frac{\pi}{4}} \tan^2 u du$$

$$= 3 \int_0^{\frac{\pi}{4}} (\sec^2 u - 1) du$$

$$= 3[\tan u - u]_0^{\frac{\pi}{4}}$$

$$= 3\left[1 - \frac{\pi}{4}\right] - (0)$$

$$= \frac{3}{4}(4 - \pi)$$